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Set-Prime Graph of a Finite Group

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Abstract

Let S be a non-empty set of positive integers. We define the set-prime graph $G_S(\Gamma)$ of a given finite group Γ of order n with respect to S, as a graph with vertex set $V(G_S(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $G_S(\Gamma)$ if and only if $(o(a), o(b)) \in S$. In this paper, we observe that order prime and general order prime graphs are special cases of set-prime graphs and we investigate some properties of set-prime graphs of finite groups.

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1 Introduction

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Herstein [3] and Harary [2] respectively. The non-standard will be given in this paper as and when required.

Throughout this paper, Γ denotes a finite group and we denote the identity element of Γ by e. The group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The order of an element a in a group Γ is denoted by $o(a)$ and order of Γ is denoted by $o(\Gamma)$. The greatest common divisor (gcd) of two numbers x and y is denoted by (x, y). The Eulers phi-function is denoted by ϕ.

In [8], M. Sattanathan and R. Kala defined the order prime graphs of finite groups and studied some properties of order prime graphs. Further, Ma et al [5] and

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Dorbidi [1] studied order prime graphs of finite groups but by calling them as coprime graphs. We defined the general order prime graphs of finite groups and studied some properties in [6, 7]. In this paper, we define set-prime graph of a finite group with respect to a set of positive integers and we observe that order prime and general order prime graphs are special cases of set-prime graphs. Further, we investigate some properties related to diameter, dominating sets, planarity and isomorphism of set-prime graphs of finite groups.

2 Definitions and Observation

Let Γ be a finite group of order n and let S be a non-empty set of positive integers. We recall the definitions of order prime graph and general order prime graph, we define set-prime graph of Γ and we list some observations.

Definition 2.1. *[8] The order prime graph $OP(\Gamma)$ of Γ is defined as a graph with the vertex set $V(OP(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $OP(\Gamma)$ if and only if $(o(a), o(b)) = 1$.*

Definition 2.2. *[6] The general order prime graph $GOP(\Gamma)$ of Γ is defined as a graph with vertex set $V(GOP(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $GOP(\Gamma)$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$.*

Definition 2.3. *The set-prime graph $G_S(\Gamma)$ of Γ with respect to S is defined as a graph with vertex set $V(G_S(\Gamma)) = \Gamma$ and any two vertices a and b are adjacent in $GOP(\Gamma)$ if and only if $(o(a), o(b)) \in S$.*

Note: We do not consider self-loops in $G_S(\Gamma)$, though we have, for some $a \in \Gamma$, $(o(a), o(a)) \in S$.

Observation: From the Definitions 2.1, 2.2 and 2.3 we have the following:

1. If $1 \notin S$, then $G_S(\Gamma)$ is a disconnected graph ($\because o(e) = 1$ in Γ).
2. If $1 \in S$, then $G_S(\Gamma)$ is connected, $deg(e) = n - 1$ and the maximum degree $\Delta(G_S(\Gamma)) = n - 1$.
3. $G_{\{1\}}(\Gamma) = OP(\Gamma)$.
4. $G_S(\Gamma) = GOP(\Gamma)$ if $S = \{1\} \cup \{p \mid p \text{ is a prime less than } n\}$.
5. If $1 \in S$, then $OP(\Gamma)$ is a subgraph of $G_S(\Gamma)$.
6. $OP(\Gamma) \subseteq G_{\{1,q\}}(\Gamma) \subseteq GOP(\Gamma)$, where q is a prime. Moreover, we have,
   (a) $OP(\Gamma) = G_{\{1,q\}}(\Gamma)$, when $q \nmid o(\Gamma)$,
   (b) $OP(\Gamma) \subsetneq G_{\{1,q\}}(\Gamma)$, when $q | o(\Gamma)$.
7. If all the positive divisors of n are in S, then $G_S(\Gamma)$ is a complete graph on n vertices.

3 Main Results

In this section we discuss some properties of set-prime graphs.

Proposition 3.1. *If Γ is a finite group and S is a set of positive integers with $1 \in S$, then $G_S(\Gamma)$ is connected and $diam(G_S(\Gamma)) \leq 2$.*

Proof. Since e is adjacent to every vertex in $G_S(\Gamma)$, it follows that given any two distinct vertices x and y in $G_S(\Gamma)$, $d(x, y) \leq 2$. Hence Γ is connected and $diam(G_S(\Gamma)) \leq 2$. □

Proposition 3.2. *Let Γ is a finite group and let S_1 and S_2 be sets of positive integers. If $S_1 \subseteq S_2$, then $G_{S_1}(\Gamma) \subseteq G_S(\Gamma)$.*

Proof. Let a and b be two vertices G_{S_1} that are adjacent in $G_{S_1}(\Gamma)$. Then $(o(a), o(b)) \in S_1$. Since $S_1 \subseteq S_2$, $(o(a), o(b)) \in S_2$ and so a and b are adjacent in $G_{S_1}(\Gamma)$. Hence $G_{S_1}(\Gamma) \subseteq G_S(\Gamma)$. □

Remark 1. Converse of the Propoition 3.2 is not true. For eg., consider the group $\mathbb{Z}_3$. The set-prime graphs $G_{\{1,3\}}(\mathbb{Z}_3)$ and $G_{\{1,5\}}(\mathbb{Z}_3)$ are same but $\{1, 3\} \not\subseteq \{1, 5\}$ and $\{1, 5\} \not\subseteq \{1, 3\}$.

Proposition 3.3. *Let Γ be a group of order n and let S be a set of positive integers with $1 \in S$. Then $\{e\}$ is a dominating set in $G_S(\Gamma)$. Moreover, if $x \in \Gamma$ and $o(x)$ is a prime number with $o(x) \in S$, then $\{x\}$ is a dominating set in $G_S(\Gamma)$.*

Proof. Since $(o(e), o(y)) = 1, \forall y \in \Gamma$, it follows that, e is adjacent to every element of Γ in $G_S(\Gamma)$, and so $\{e\}$ is a dominating set in $G_S(\Gamma)$.

If $x \in \Gamma$ and $o(x) = p$, a prime with $p \in S$, then $(o(x), o(y)) = 1$ or p, $\forall y \in \Gamma$. Hence x is adjacent to every element of Γ in $G_S(\Gamma)$, and so $\{x\}$ is a dominating set in $G_S(\Gamma)$. □

Theorem 3.4. *Let S be a set of positive integers with 1. If Γ is a finite group of prime order q, then*

$$G_S(\Gamma) = \begin{cases} OP(\Gamma) & \text{if } q \notin S \\ K_q & \text{if } q \in S \end{cases}.$$

Proof. If $q \notin S$, then $(o(a), o(b)) \notin S - \{1\}$ for any $a, b \in \Gamma$ (otherwise there exist a positive integer greater $m \notin \{1, q\}$ such that $m \mid q$, which is not possible). Hence $a, b \in \Gamma$ are adjacent in $G_S(\Gamma)$ if and only if $(o(a), o(b)) = 1$. Therefore $G_S(\Gamma) = OP(\Gamma)$ if $q \notin S$.

If $q \in S$, then for $a, b \in \Gamma$, $(o(a), o(b)) = 1$ or q so that $(o(a), o(b)) \in S$. Hence every vertex is adjacent to every other vertex in $G_S(\Gamma)$. Therefore $G_S(\Gamma)) = K_q$ if $q \notin S$. □

Corollary 3.5. *Let Γ be a group of prime order > 2 and let S bs a set of positive integers with $1 \in S$. If $o(\Gamma) \notin S$, then $\{e\}$ is the unique dominating set in $G_S(\Gamma)$.*

Proof. Suppose that $o(\Gamma) \notin S$. Since Γ is a group of prime order, by the Theorem 3.4, $G_S(\Gamma) = OP(\Gamma)$. By the Proposition 2.2 in [5], $\{e\}$ is the unique dominating set in $OP(\Gamma)$. Thus it follows that $\{e\}$ is the unique dominating set in $G_S(\Gamma)$. □

Theorem 3.6. *Let p be a prime number. If Γ is a finite cyclic group of composite order n and $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$, then $p \nmid n$.*

Proof. Let if posiible $p|n$. By Cauchy's theorem for finite groups, there exists an element $a \in \Gamma$ with $o(a) = p$. Since Γ is a cyclic group of order n, there exists an element $b \in \Gamma$ with $o(b) = n$. Now $(o(a), o(b)) = p$ which is a contradiction (because a and b are adjacent $G_{\{1,p\}}(\Gamma)$ and they are not adjacent in $OP(\Gamma)$, but $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$). So $p \nmid n$. □

Corollary 3.7. *Let p be a prime number and Γ be a finite cyclic group. Then $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$ if and only if $p \nmid o(\Gamma)$.*

Proof. Follows from Theorems 3.4 and 3.6 and our observation (4) in Section 2. □

Theorem 3.8. *Let Γ be a finite group of order $n > 2$. If $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$, for any prime p with $p < n$, then n is a prime number (and hence Γ is a cyclic group).*

Proof. Suppose that $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$, for any prime p with $p < n$. Let if possible n is not a prime number then there exist a prime number q dividing n. Then by our observation (4) in section 2, we have, $OP(\Gamma) \subsetneq G_{\{1,q\}}(\Gamma)$, which is not possible. Hence $o(\Gamma)$ must be a prime number. □

Corollary 3.9. *An integer $n > 2$, is a prime if and only if $OP(\mathbb{Z}_n) = G_{\{1,p\}}(\mathbb{Z}_n)$, for every prime p with $p < n$.*

Proof. Follows from Theorem 3.8 and Corollary 3.7. □

Theorem 3.10. *Let Γ be a group of finite order $n > 2$. Then $G_{\{1,p\}}(\Gamma) \cong K_{1,n-1}$, for any prime p with $p < n$ if and only if n is a prime number.*

Proof. Suppose that n is a prime number. Then Γ is a cyclic group. Let $\Gamma = \{e, a, a^2, \ldots, a^{n-1}\}$, where e is the identity element and a is a generator of the group Γ. Note that in the group Γ, $o(e) = 1$ and $o(a^i) = n$, $1 \leq i \leq n-1$. Hence $(o(e), o(a^i)) = 1$ and $(o(a^i), o(a^j)) = n$, $1 \leq i, j \leq n-1$. Therefore e is adjacent to a^i for all $i = 1, 2, \ldots, n-1$ and a^is are mutually non-adjacent in $G_{\{1,p\}}(\Gamma)$, for every prime p with $p < n$. Hence $G_{\{1,p\}}(\Gamma) \cong K_{1,n}$, for every prime p with $p < n$.

Conversely, suppose that $G_{\{1,p\}}(\Gamma) \cong K_{1,n-1}$, for any prime p with $p < n$. Clearly, $G_{\{1,p\}} - e$ is totally disconnected for any prime p with $p < n$. We claim that $o(\Gamma) = n$ is a prime number. If n is not a prime number, there exist a prime q with $q < n$ such that $q|n$. Since $p \mid o(\Gamma)$, by the Cauchy's theorem for finite groups, there exists an element a in Γ such that $o(a) = q$. Let us consider a^2 $(\neq e)$ in Γ. Note that $o(a^2) = q$ and $(o(a), o(a^2)) = q$. Hence a and a^2 are adjacent in $G_{\{1,p\}} - e$, which is a contradiction and so n is a prime number. □

The following corollaries are immediate from the Theorem 3.10:

Corollary 3.11. *Let Γ be a finite group of order $n > 2$. Then $G_{\{1,p\}}(\Gamma)$ is a tree, for any prime p with $p < n$ if and only if $o(\Gamma)$ is a prime number.*

Corollary 3.12. *An integer $n > 2$ is prime if and only if $G_{\{1,p\}}(\mathbb{Z}_n)$ is a tree, for any prime p with $p < n$.*

Corollary 3.13. *If Γ is a group of prime order $q > 2$, then $G_{\{1,p\}}(\Gamma)$ is planar, for any prime p with $p < q$.*

Corollary 3.14. *Let Γ be a finite group of prime order q and let p be a prime. Then*

$$diam(G_{\{1,p\}}(\Gamma)) = \begin{cases} 1, & \text{when } q = 2 \\ 2, & \text{when } q > 2 \text{ and } p < q, \\ 1, & \text{when } q > 2 \text{ and } p = q, \\ 2, & \text{when } q > 2 \text{ and } p > q. \end{cases}$$

Proof. If $q = 2$, then $\Gamma \cong \mathbb{Z}_2$ and so $diam(G_{\{1,p\}}(\Gamma)) = 1$.

If $q > 2$, then by Theorem 3.10, it follows that $G_{\{1,p\}}(\Gamma) \cong K_{1,q-1}$, $\forall p < n$. So $diam(G_{\{1,p\}}) = 2$ when $p < q$.

If $q > 2$ and $p = q$, $G_{\{1,p\}}(\Gamma) \cong K_p$. So $diam(G_{\{1,p\}}(\Gamma)) = 1$.

If $q > 2$ and $p > q$, then $G_{\{1,p\}}(\Gamma) = OP(\Gamma)$. So $diam(G_{\{1,p\}}(\Gamma)) = 2$. □

Proposition 3.15. *Let Γ be a group with $o(\Gamma) = pq$, where p and q are distinct primes. Then*

$$diam(G_{\{1,p,q\}}(\Gamma)) = \begin{cases} 2, & \text{when } \Gamma \text{ is cyclic} \\ 1, & \text{when } \Gamma \text{ is non-cyclic.} \end{cases}$$

Proof. We consider two cases: (i) when Γ is cyclic and (ii) when Γ is non-cyclic.
Case (i): If Γ is cyclic, then there are $\phi(pq) = (p-1)(q-1)$ number of generators. Since p and q are distinct, $(p-1)(q-1) \geq 2$ and so there are at least two generators x, y in Γ. Note that $o(x) = o(y) = pq$ and so $(o(x), o(y)) \neq 1$ or p or q. Hence x and y are not adjacent in $G_{\{1,p,q\}}(\Gamma)$. Since e is adjacent to both both x and y in $G_{\{1,p,q\}}(\Gamma)$, it follows that $diam(G_{\{1,p,q\}}(\Gamma)) = 2$.
Case (ii): If Γ is non-cyclic, then the order of any element other than e is either p or q, and so for any two distinct elements $x, y \in \Gamma$, $(o(x), o(y)) = 1$ or p or q. Hence x and y are adjacent in $G_{\{1,p,q\}}(\Gamma)$ and it follows that $diam(G_{\{1,p,q\}}(\Gamma)) = 1$. □

Proposition 3.16. *Let Γ be a group with $o(\Gamma) = p^2$, where p is a prime. Then*

$$diam(G_{\{1,p\}}(\Gamma)) = \begin{cases} 2, & \text{when } \Gamma \text{ is cyclic} \\ 1, & \text{when } \Gamma \text{ is non-cyclic.} \end{cases}$$

Proof. Follows in similar lines as in the proof of Proposition 3.15. □

By our observation (5) in section 2, if S is the set of all prime integers less than $o(\Gamma)$ with $1 \in S$, then $G_S(\Gamma) = GOP(\Gamma)$. Hence we have the following result:

Theorem 3.17. *[7] Let Γ be a group of finite order n and let S be the set of all prime integers less than n with $1 \in S$. We have*

1. *n is a prime if and only if $G_S(\Gamma) \cong K_{1,n-1}$.*

2. *n is a prime if and only if $G_S(\Gamma)$ has a pendant vertex.*

3. *$G_S(\Gamma)$ has precisely two pendant vertices if and only if Γ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proposition 3.18. *If Γ is a group of order 4, then $G_{\{1,2\}}(\Gamma)$ is planar.*

Proof. If Γ is a group of order 4, then $\Gamma \cong \mathbb{V}_4$, the Klein-4 group or $\Gamma \cong \mathbb{Z}_4$. Hence $G_{\{1,2\}}(\Gamma)$ is isomorphic to K_4, the complete graph on 4 vertices or a subgraph of K_4. In any case $G_{\{1,2\}}(\Gamma)$ is planar. □

Proposition 3.19. *Let Γ be a finite group such that $o(\Gamma)$ is divisible by two distinct primes p and q. Then $G_{\{1,p,q\}}(\Gamma)$ is non-planar.*

Proof. By the Cauchy's theorem for finite grups, there exist two elements $x, y \in \Gamma$ such that $o(x) = p$ and $o(y) = q$. Then by the definition of set-prime graph, it follows that, e, x and y are adjacent to every vertex in $G_{\{1,p,q\}}(\Gamma)$. Since p and q are distinct, $o(\Gamma) \geq 6$ and $G_{\{1,p,q\}}(\Gamma)$ contains a copy of $K_{3,3}$. Hence $G_{\{1,p,q\}}(\Gamma)$ is non-planar. □

Corollary 3.20. *Let Γ be a finite group such that $o(\Gamma)$ is divisible by two distinct primes p and q. If $1, p, q \in S$, then $G_S(\Gamma)$ is non-planar.*

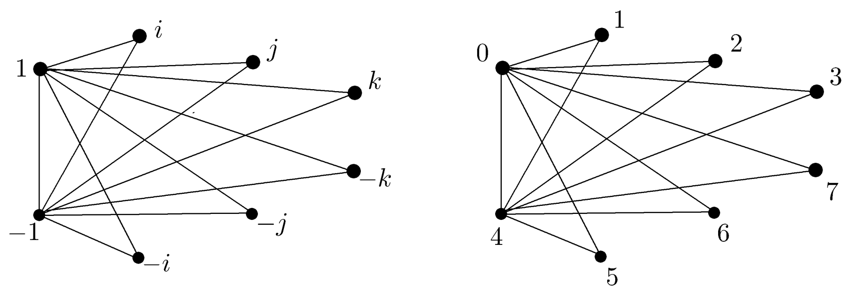
Proof. Follows by Propositions 3.2 and 3.19. □

Proposition 3.21. *Let Γ be a finite group. If $o(\Gamma) = p^2$, where p is an odd prime, then $G_{\{1,p\}}(\Gamma)$ is non-planar.*

Proof. We consider two cases: (i) when Γ is cyclic and (ii) when Γ is non-cyclic.
Case (i): If Γ is cyclic, then there are $\phi(p^2) = p^2 - p$ number of generators. The number of elements other than generators is $p^2 - (p^2 - p) = p \geq 3$. Thus there are at least three elemnts whose orders are 1 or p, and these elements are adjacent to every element in $G_{\{1,p\}}(\Gamma)$. Since $o(\Gamma) = p^2 \geq 9$, it follows that $G_{\{1,p\}}(\Gamma)$ contains a copy of $K_{3,3}$ and so $G_{\{1,p\}}(\Gamma)$ is non-planar.
Case (ii): If Γ is non-cyclic, then every element in Γ is of order 1 or p. Hence any two distinct vertices are adjacent in $G_{\{1,p\}}(\Gamma)$. Since $o(\Gamma) = p^2 \geq 9$, it follows that $G_{\{1,p\}}(\Gamma)$ contains a copy of K_5 and so $G_{\{1,p\}}(\Gamma)$ is non-planar. □

Proposition 3.22. *Let Γ_1 and Γ_2 be two finite groups and let S be a non-empty set of positive integers. If $\Gamma_1 \cong \Gamma_2$, then $G_S(\Gamma_1) \cong G_S(\Gamma_2)$.*

Proof. Let $\eta : \Gamma_1 \to \Gamma_2$ be a group isomorphism. Clearly, η is a bijective mapping of $V(G_S(\Gamma_1))$ onto $V(G_S(\Gamma_2))$. Let x and y be two vertices in $G_S(\Gamma_1)$. Since $o(a) = o(\eta(a)), \forall a \in \Gamma$, it follows that, $xy \in E(G_S(\Gamma_1))$ if and only if $\eta(x)\eta(y) \in E(G_S(\Gamma_2))$. Thus η is a graph isomorphism of $G_S(\Gamma_1)$ onto $G_S(\Gamma_2)$. $\square$



$G_{\{1,2\}}(\mathbb{H}_8)$ $G_{\{1,2\}}(\mathbb{Z}_8)$

Fig 1.

Remark 2. The converse of the Proposition 3.22 is not true in general. For eg., consider the groups $\mathbb{Z}_8$, the group of residue classes modulo 8 and $\mathbb{H}_8$, the group of quaternion units. In $\mathbb{Z}_8$, $o(0) = 1, o(1) = 8, o(2) = 4, o(3) = 8, o(4) = 2, o(5) = 8, o(6) = 4, o(7) = 8$. In $\mathbb{H}_8$, $o(1) = 1, o(-1) = 2, o(i) = 4, o(-i) = 4, o(j) = 4, o(-j) = 4, o(k) = 4, o(-k) = 4$. The graphs $G_{\{1,2\}}(\mathbb{Z}_8)$ and $G_{\{1,2\}}(\mathbb{H}_8)$ are the same except for the labeling (shown in Fig 1.). Hence $G_{\{1,2\}}(\mathbb{Z}_8)$ and $G_{\{1,2\}}(\mathbb{H}_8)$ are isomorphic as graphs, but $\mathbb{Z}_8$ and $\mathbb{H}_8$ are not isomorphic as groups.

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